Coding theory in post-quantum cryptography

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Funded by Romanian Governement:
PN-III-P1-1.1-PD-2019-0285 CodebasedCrypto

Modern cryptography [DH78]

Modern Cryptography

A pair $(\mathbf{sk}, \mathbf{pk})$ s.t.

sk	$\rightsquigarrow$	**pk**	easy	
sk	$\hookleftarrow$	**pk**	difficult	RSA ('78), El Gamal ('85)

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The difficulty of the mathematical problems [1] [2]

1. 2014. R. Barbulescu, P. Gaudry, A. Joux, and E. Thomé. "A heuristic quasi-polynomial algorithm for discrete logarithm in finite fields of small characteristic."
   2018. R. Granger, T. Kleinjung, and J. Zumbragel. "On the discrete logarithm problem in finite fields of fixed characteristic".
2. 1997. Peter W. Shor. "Polynomial-time algorithms for prime factorization and

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The difficulty of the mathematical problems

NIST – post-quantum cryptography project [1]

- Hash based cryptography
- Lattice based cryptography
- Code based cryptography
- Multivariate cryptography

1. http ://csrc.nist.gov/groups/ST/post-quantum-crypto/

Error correcting codes

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Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$

$$\|\boldsymbol{x}\|_H \stackrel{def}{=} |\{i \mid x_i \neq 0\}| \quad \mathrm{d_H}(\boldsymbol{x}, \boldsymbol{y}) \stackrel{def}{=} |\{i \mid x_i \neq y_i\}|$$

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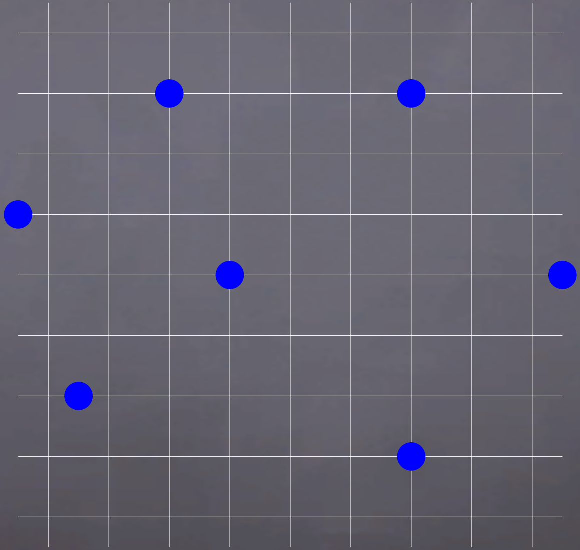
Example

Let $q = 2$ and $\boldsymbol{x} = (1, 0, 0, 1, 0)$, $\boldsymbol{y} = (1, 0, 0, 1, 1)$.
Then

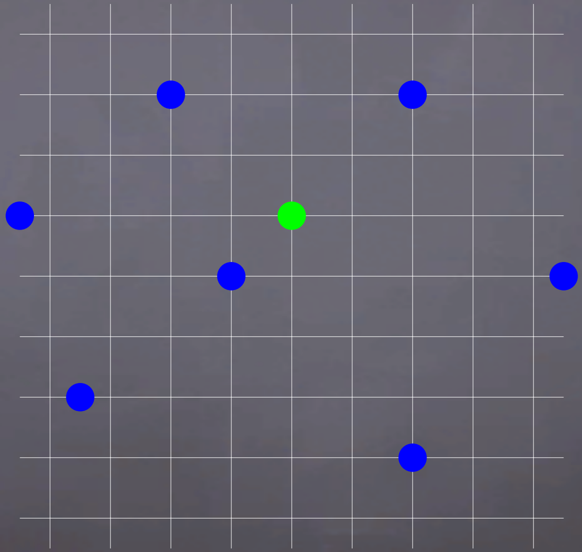
$$\|\boldsymbol{x}\|_H = 2 \quad \text{and} \quad \mathrm{d_H}(\boldsymbol{x}, \boldsymbol{y}) = 1$$

Complete decoding

Complete decoding



Complete decoding



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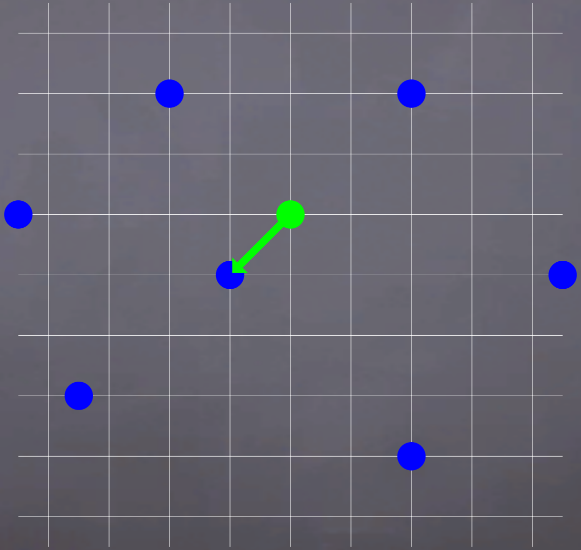

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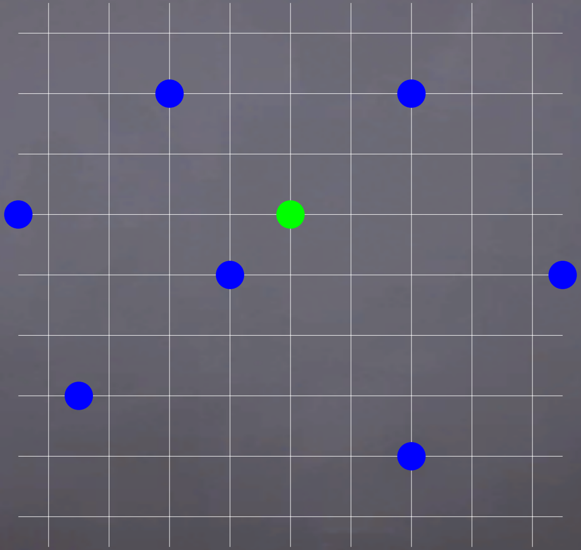

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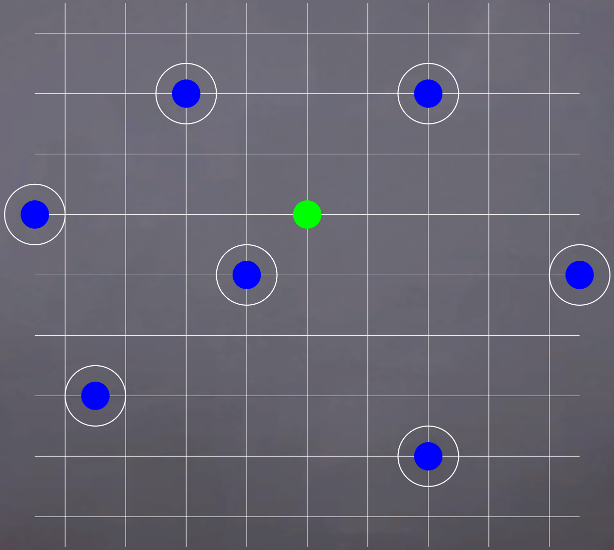

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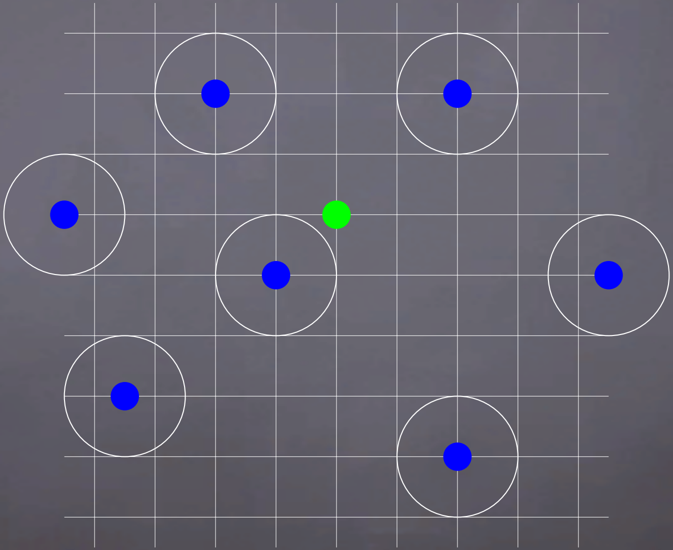

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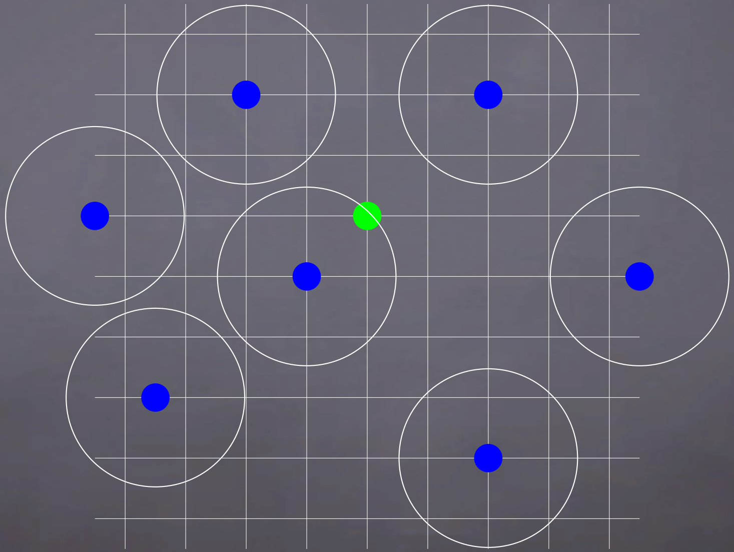

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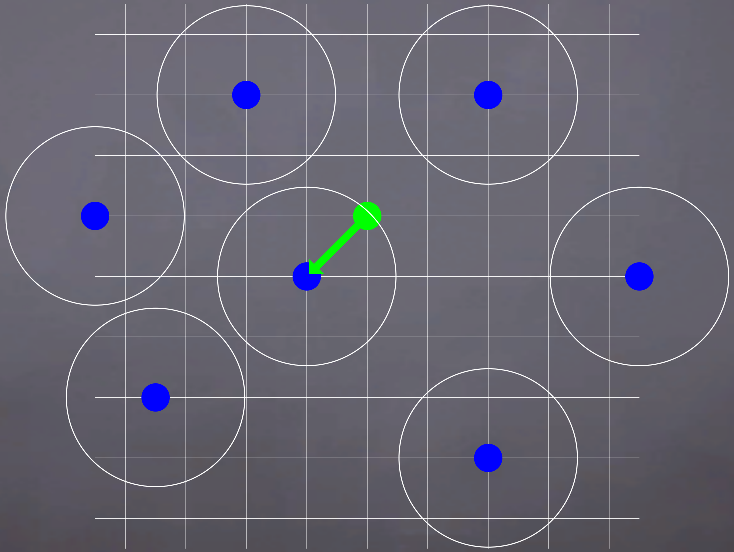
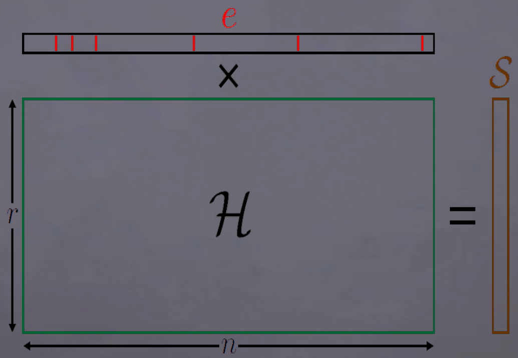
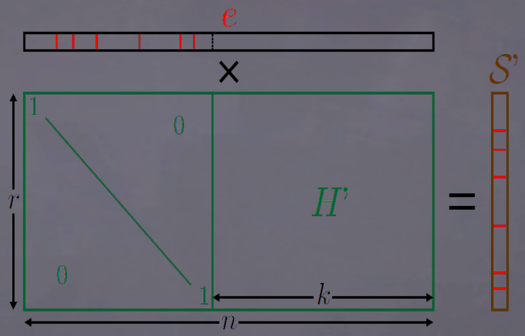


Syndrome decoding [2]



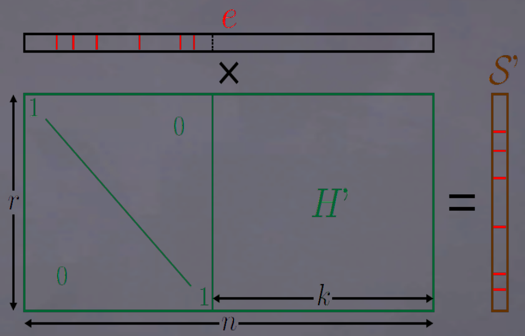
2. 1978. Berlekamp E., McEliece R.J., Van Tilborg "On the inherent intractability of certain coding problems."

Information Set Decoding (ISD) [3]



3. Prange(1957), Stern(1988), Dumer (1991), Canteaut and Chabaud (1998), May, Meurer and Thomae (2011), Becker, Joux, May and Meurer (2012), May and Ozerov (2015)

Information Set Decoding (ISD) [4]



Complexity of ISD [3] for $\|e\| = o(n)$ is roughly

$$2^{c\|e\|(1+o(1))}.$$

3. 2016. Canto-Torres and Sendrier - "Analysis of Information Set Decoding for a Sub-linear Error Weight".
4. Prange(1957), Stern(1988), Dumer (1991), Canteaut and Chabaud (1998), May, Meurer and Thomae (2011), Becker, Joux, May and Meurer (2012), May and Ozerov (2015)

McEliece cryptosystem ('78)

1. Main idea
   - The private key = a code $\mathscr{C}$ with an efficient decoding algorithm
   - The public key = a random basis for $\mathscr{C}$
2. McEliece propose to use binary Goppa codes.

McEliece cryptosystem

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    3. The private key sk $= (\boldsymbol{S}, \boldsymbol{G}, \boldsymbol{P})$ and the public key pk $= (\boldsymbol{G}_{pub}, t)$ with

$$\boldsymbol{G}_{pub} = \boldsymbol{SGP}$$

McEliece cryptosystem

Encryption

Let $\boldsymbol{m} \in \mathbb{F}_q^k$,

1. Generate a random error vector $\boldsymbol{e} \in \mathbb{F}_q^n$ of Hamming weight t
2. Encrypt $\boldsymbol{c} = \boldsymbol{m}\boldsymbol{G}_{pub} + \boldsymbol{e}$

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Decryption

1. Compute $\boldsymbol{z} = \boldsymbol{c}\boldsymbol{P}^{-1}$ $\qquad \boldsymbol{z} = \boldsymbol{m}\boldsymbol{S}\boldsymbol{G} + \boldsymbol{e}\boldsymbol{P}^{-1}$
2. Compute $\boldsymbol{y} = Decode_{\boldsymbol{G}}(\boldsymbol{z})$ $\qquad \boldsymbol{y} = \boldsymbol{m}\boldsymbol{S}$
3. Return $\boldsymbol{m}' = \boldsymbol{y}\boldsymbol{S}^{-1}$ $\qquad \boldsymbol{m}' = \boldsymbol{m}$

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7. 2013. Faugère, Gauthier, Otmani, Perret and Tillich. - "A distinguisher for high rate McEliece cryptosystems"

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1. **Increase the minimum distance**
2. **Add extra structure (quasi-cyclic, quasi-dyadic)**
3. Change the metric (Rank)

Classic variants	*Proposal*
Binary Goppa	1978 [McE78]
GRS	1986 [Nie86]
Reed-Muller	1994 [Sid94]
Concatenated	1994 [Sen94]
Algebraic geometric	1996 [JM96]
Wild Goppa	2010 [BLP10]
Convolutional	2012 [LJ12]
Polar	2014 [SK14]

QC, QD Variants	
QC-BCH	2005 [Gab05]
QC-LDPC	2008 [BBC08]
QC-Alternant	2009 [BCGO09]
QD-Goppa	2009 [MB09]
QD-Srivastava	2012 [Per12]
QC-MDPC	2012 [MTSB13]

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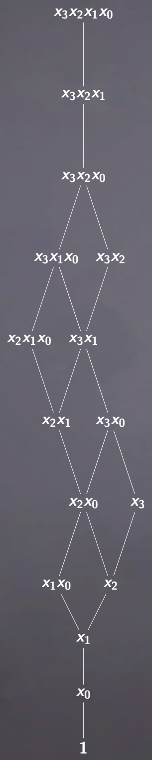
QC-MDPC	*2012 [MTSB13]*	*2016 [BDLO16]*



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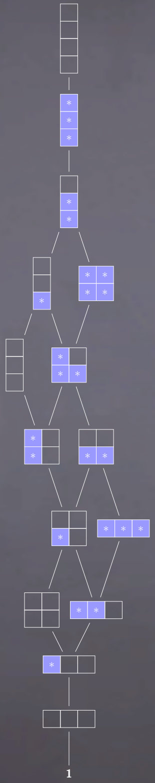

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Questions ?

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