On the Roots of Certain Reliability Polynomials

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Reliability

It comes as no surprise that hundreds of seemingly natural definitions arise by examining the plethora of different types of networks, causes and types of failures, and levels and types of operation. One should not expect to find a single definition for reliability that accommodates the many real situations of importance.

(Charles J. Colbourn)

Reliability

- Colbourn, Barlow and Proschan, Ball and Provan – Network theory

  The connectivity of the input or the source with the output or the terminus

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  The connectivity of the input or the source with the output or the terminus

- Moore and Shannon – Information theory

  Connectivity and non-connectivity

  A large improvement in reliability, both when the coil is energized and when it is not energized

  (Moore and Shannon)

Two-terminal networks/circuits

$$\boldsymbol{N} = \{\mathcal{D}, \mathcal{P}, \mathcal{W}, \{S, T\}\}$$

$n = 6$

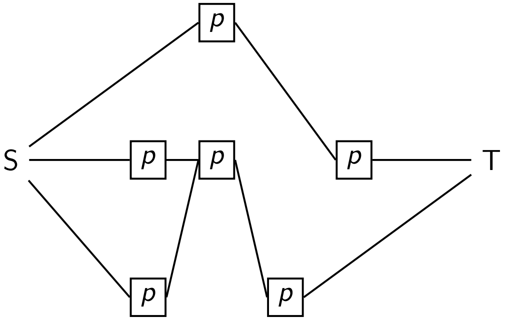
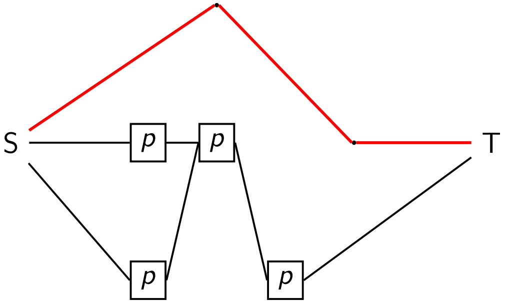


Figure – Two-terminal network.

Two-terminal networks/circuits

$$\boldsymbol{N} = \{\mathcal{D}, \mathcal{P}, \mathcal{W}, \{S, T\}\}$$

$n = 6 \qquad l = 2$



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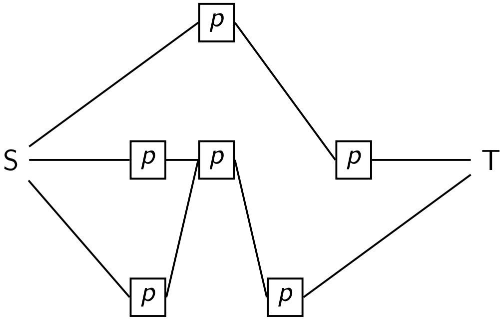

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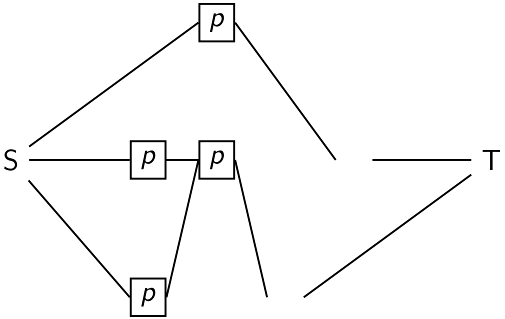

FIGURE – Two-terminal network.

Minimal networks

Definition 1

Any two-terminal network that satisfies the following condition is called minimal

$$n = lw \tag{1}$$

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$$n = lw \quad (1)$$

$l = 4, w = 3$ allowed

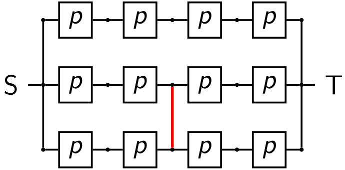

FIGURE – Matchstick minimal networks.

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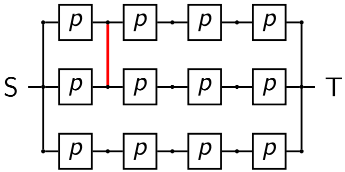

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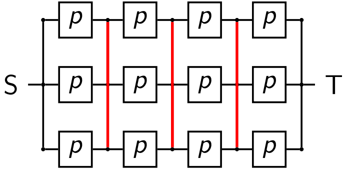

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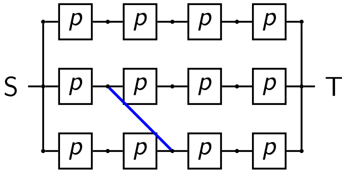
Not allowed



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Minimal networks

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Any two-terminal network that satisfies the following condition is called minimal

$$n = lw \tag{1}$$

Not allowed $\quad l = 3, w = 3$

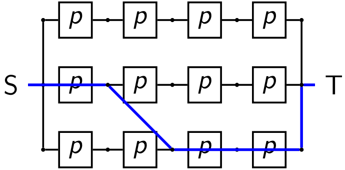

Figure – Matchstick minimal networks.

Compositions of series and parallel

- The basic elements are

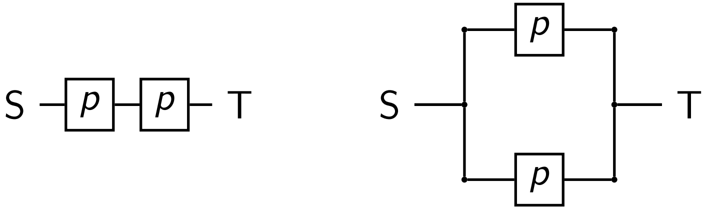

FIGURE – The basic compositions $\boldsymbol{C}^{(0)}$ (left) and $\boldsymbol{C}^{(1)}$ (right).

Compositions of series and parallel

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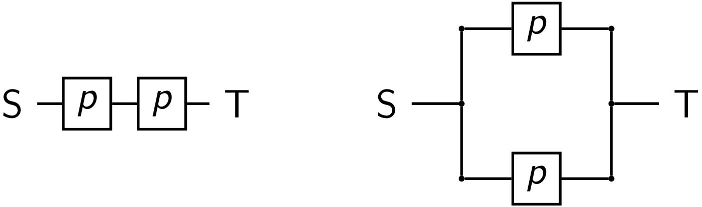
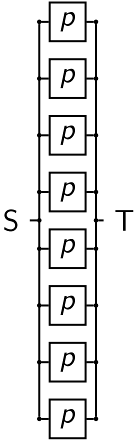


FIGURE – The basic compositions $\boldsymbol{C}^{(0)}$ (left) and $\boldsymbol{C}^{(1)}$ (right).

- Any binary vector of length m gives a composition of series and parallel with $n = 2^m$.

Compositions of series and parallel

$\boldsymbol{C}^{(0,0,0)}$



$\boldsymbol{C}^{(1,1,1)}$

Compositions of series and parallel

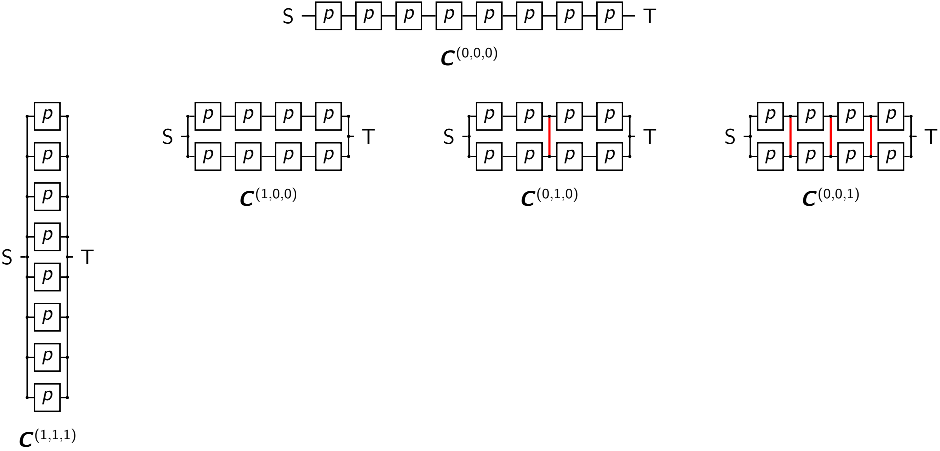


FIGURE – The elements of the set $\mathcal{C}_{2^3}$.

Compositions of series and parallel

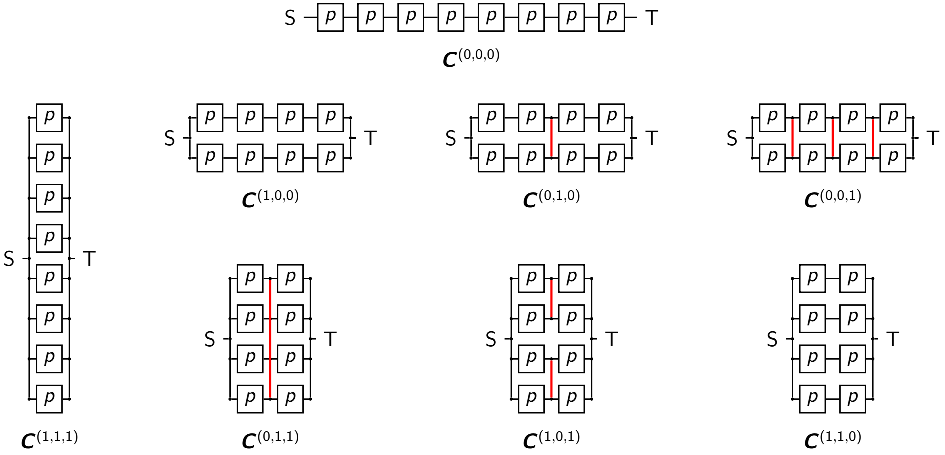


FIGURE – The elements of the set $\mathcal{C}_{2^3}$.

Reliability polynomials

Theorem 2

Let m be a strictly positive integer and $\boldsymbol{u} = (u_0, \ldots, u_{m-1}) \in \{0,1\}^m$. Then :

$$\mathrm{Rel}(\boldsymbol{C}^{\boldsymbol{u}}; p) = \mathrm{Rel}(\boldsymbol{C}^{(u_0)}) \circ \cdots \circ \mathrm{Rel}(\boldsymbol{C}^{(u_{m-1})}; p), \tag{2}$$

where $\mathrm{Rel}(\boldsymbol{C}^{(0)}; p) = p^2$ *and* $\mathrm{Rel}(\boldsymbol{C}^{(1)}; p) = 1 - (1-p)^2$.

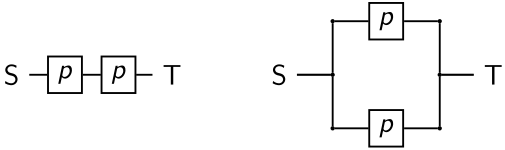


Figure – The basic compositions $\boldsymbol{C}^{(0)}$ (left) and $\boldsymbol{C}^{(1)}$ (right).

Roots of two-terminal networks

- *Heading out the highway* – 2020 survey in network reliability points out new directions (analytical properties)
- Understanding the nature of the complex roots could lead to other fundamental discoveries (optimality)
- Complex roots - recent topic, not-well understood (conjectures dispoved, some special cases were investigates) , no algorithm for computing the roots

Generating the Roots Efficiently

Theorem 3

If $P(x) = \prod_{k=1}^{l}(x - \alpha_k)$ *is a polynomial of degree* l*, having the complex roots* α_k*,* $k = 1, \dots, l$*, then*

- $P(\mathrm{Rel}(\boldsymbol{C}^{(0)}; p))$ *has* $2l$ *complex roots* $\{\sqrt{\alpha_k}, -\sqrt{\alpha_k}\}_{1 \le k \le l}$.
- $P(\mathrm{Rel}(\boldsymbol{C}^{(1)}; p))$ *has* $2l$ *complex roots* $\{1 - \sqrt{1 - \alpha_k}, 1 + \sqrt{1 - \alpha_k}\}_{1 \le k \le l}$.

For $m = 1$ we can see that

- $\mathrm{Rel}(\boldsymbol{C}^{(0)}; p) = p^2$ has 0 as double root ;
- $\mathrm{Rel}(\boldsymbol{C}^{(1)}; p) = 1 - (1 - p)^2$ has the simple roots $0, 2$.

Generating the Roots Efficiently

Algorithm 1 Generating the roots

Require: $\boldsymbol{u} \in \{0,1\}^m$
Ensure: The list of L of all complex roots of $\mathrm{Rel}(\boldsymbol{C}^{\boldsymbol{u}}; p)$.

$g_0 = [\sqrt{x}, -\sqrt{x}], g_1 = [1-\sqrt{1-x}, 1+\sqrt{1-x}]$
$L = [0]$ {Initialization phase}
$S = []$
for $k \leftarrow 1, m$ **do**
  for $j \leftarrow 1, |L|$ **do**
    $S = \mathrm{Append}(S, \mathrm{Elem}(\mathrm{eval}(g_{u_l}, L[j])))$
  end for
  $L = S$
end for
return L

Soundness and performance

Proposition 1

Let m be a positive integer and $\boldsymbol{u} \in \{0,1\}^m$. Algorithm 1 outputs the list of all complex roots of $\mathrm{Rel}(\boldsymbol{C}^{\boldsymbol{u}}; p)$.

Proposition 2

The time complexity of Algorithm 1 is $O(n)$.

Examples

1. For $m = 1$ we have $\mathrm{eval}(g_k, 0) = \begin{cases} \{0,0\} & , \ k = 0 \\ \{0,2\} & , \ k = 1 \end{cases}$.
   These values correspond to the roots of $\mathrm{Rel}(\boldsymbol{C}^{(0)}; p)$ and $\mathrm{Rel}(\boldsymbol{C}^{(1)}; p)$.

2. For $m = 2$ we need first to compute
   $\mathrm{eval}(g_k, 2) = \begin{cases} \{\sqrt{2}, -\sqrt{2}\} & , \ k = 0 \\ \{1-i, 1+i\} & , \ k = 1 \end{cases}$.
   Therefore the roots of all four (2^2) compositions are :

$$\begin{cases} \{0,0,0,0\} & , \ \boldsymbol{u} = (0,0) \\ \{0,0,\sqrt{2},-\sqrt{2}\} & , \ \boldsymbol{u} = (1,0) \\ \{0,0,2,2\} & , \ \boldsymbol{u} = (0,1) \\ \{0,2,1-i,1+i\} & , \ \boldsymbol{u} = (1,1) \end{cases} .$$

Examples

2. For $m = 3$ we have

$$
\begin{cases}
\{0,0,0,0,0,0,0,0\} & , \boldsymbol{u} = (0,0,0) \\
\left\{0,0,0,0,\sqrt{2},\sqrt{2},-\sqrt{2},-\sqrt{2}\right\} & , \boldsymbol{u} = (0,1,0) \\
\{0,0,0,0,2,2,2,2\} & , \boldsymbol{u} = (0,0,1) \\
\{0,0,2,2,1-i,1-i,1+i,1+i\} & , \boldsymbol{u} = (0,1,1) \\
\left\{0,0,0,0,\sqrt[4]{2},-\sqrt[4]{2},i\sqrt[4]{2},-i\sqrt[4]{2}\right\} & , \boldsymbol{u} = (1,0,0) \\
\left\{0,0,2,2,1 \pm i\sqrt{\sqrt{2}-1},1 \pm \sqrt{\sqrt{2}+1}\right\} & , \boldsymbol{u} = (1,0,1) \\
\left\{0,0,\sqrt{2},-\sqrt{2},\sqrt{\frac{\sqrt{2}+1}{2}} \pm i\sqrt{\frac{\sqrt{2}-1}{2}},-\sqrt{\frac{\sqrt{2}+1}{2}} \pm i\sqrt{\frac{\sqrt{2}-1}{2}}\right\} & , \boldsymbol{u} = (1,1,0) \\
\left\{0,2,1-i,1+i,1 \pm \frac{1}{\sqrt{2}}(1-i),1 \pm \frac{1}{\sqrt{2}}(1+i)\right\} & , \boldsymbol{u} = (1,1,1)
\end{cases}
.
$$

Roots

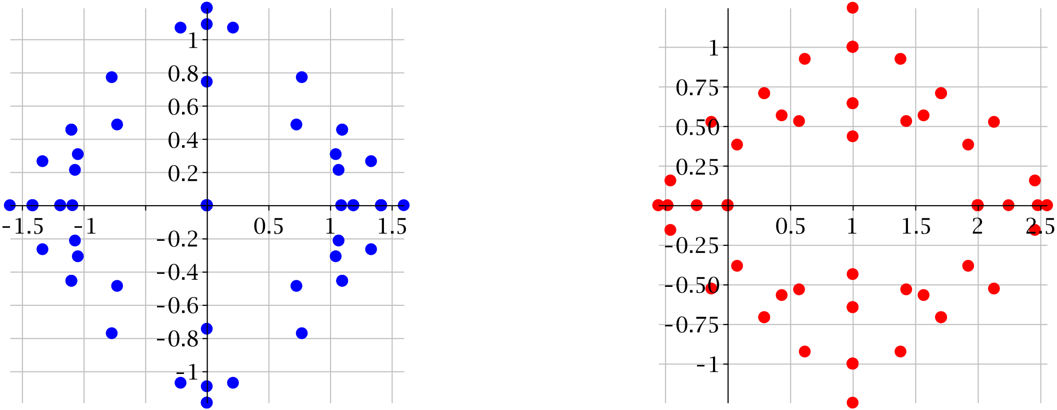


FIGURE – n= 16

Roots

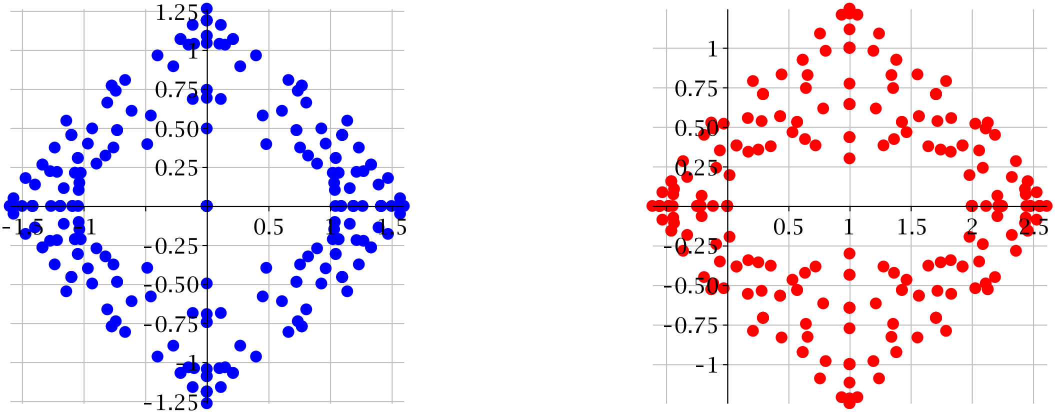


FIGURE – $n = 32$

Roots

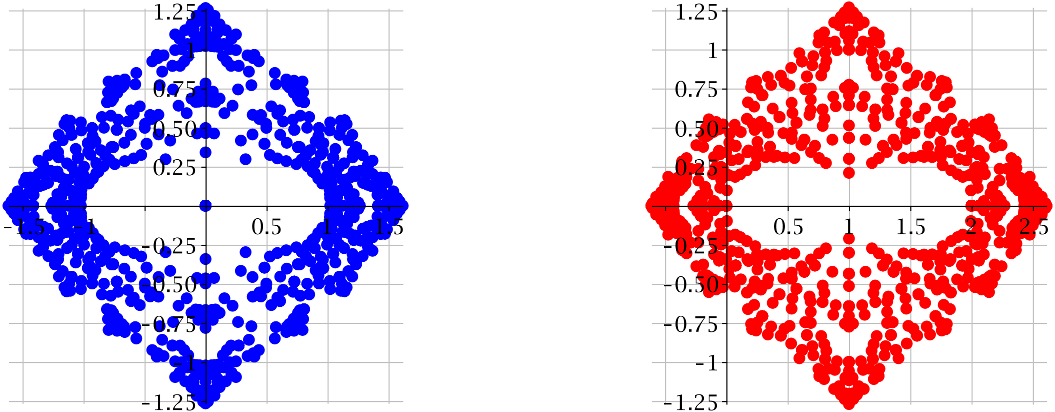
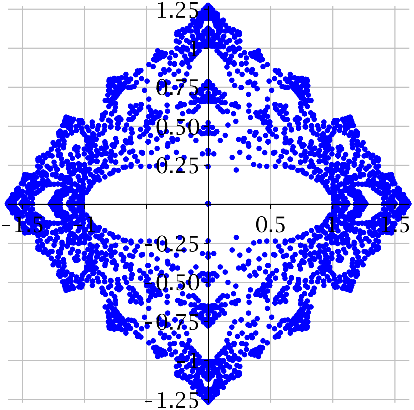


FIGURE – $n = 64$

Roots



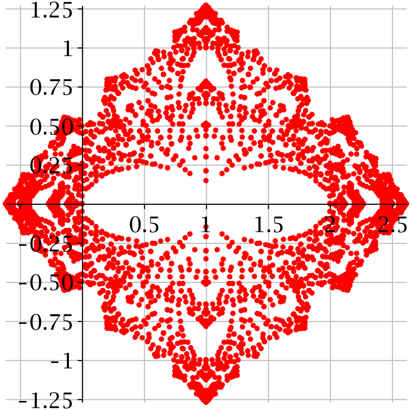
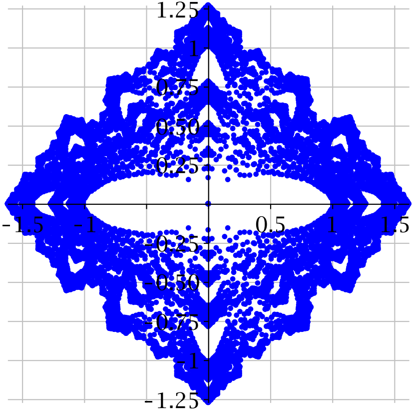


FIGURE – $n = 128$

Roots



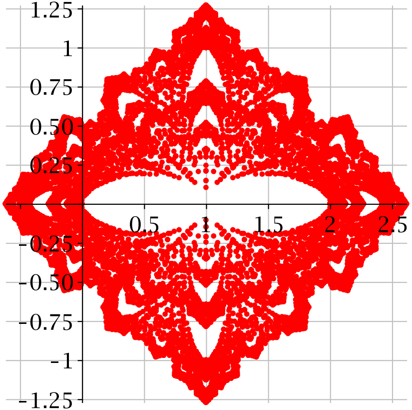
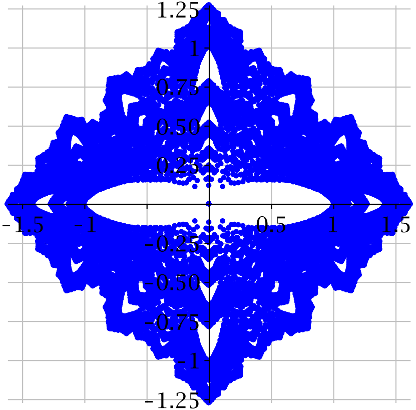


FIGURE – $n = 256$

Roots



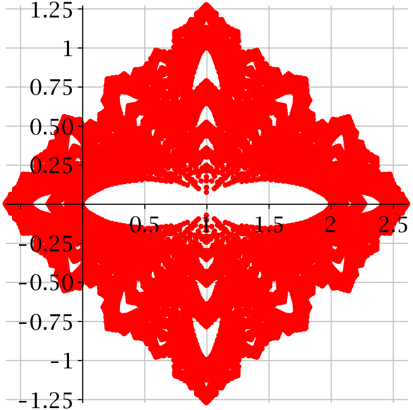
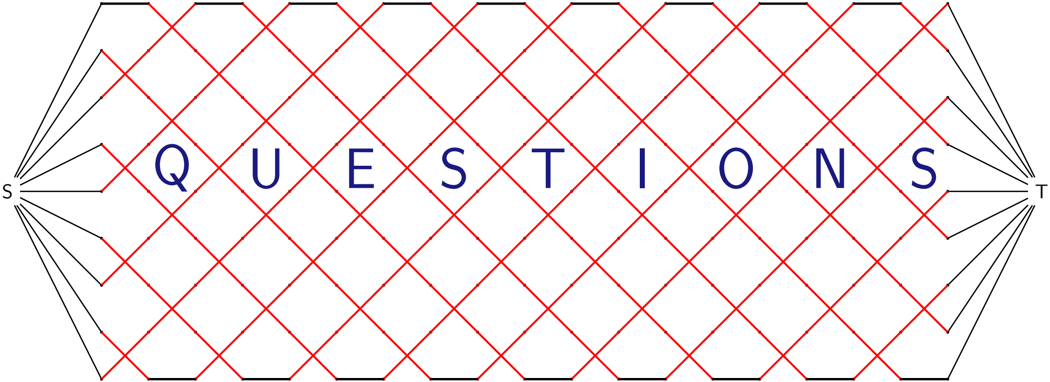


FIGURE – $n = 512$

Conclusions

- We have proposed an algorithm that efficiently computes the roots of the reliability polynomial of any composition of series and parallel
- The algorithm does not require preliminary computation of the polynomial
- Implementation optimizations could be proposed (recursive formula, symmetry properties)
- Structural properties and limit shape of the set of roots to be analyzed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